Auction Theory: Some Basics

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Outline

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- Example: uniform distribution (equal chance) in some range.

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- Bidders are playing a game of incomplete information. They will bid *below* their valuation.
- Bayes-Nash equilibrium: $b_i = \frac{n-1}{n} v_i$. (Assumption: valuations are independently drawn and distributed uniformly over the same interval).

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- Suppose bidder's valuation is 100. Believes that the second-highest bid will be below 100, say 80. Bidding truthfully gives a surplus of 20. Any bid above 80 will give the game surplus while bidding below 80 will give zero. Similar argument if she believes that the second-highest bid will be above 100.

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- Suppose the valuation of the bidders are 80 and 30. FP auction: bidders bid 40 and 15 - revenue is 40. SP auction: bidders bid 80 and 30; higher bidder wins and pays 30. FP better.

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- On average (i.e. in expectation)?

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- RE Theorem: Two auctions that generate the same outcomes in equilibrium and where losers don't pay generate the same *expected revenue*.
- FP and SP auctions lead to the same outcome in equilibrium - the highest valuation bidder gets the outcome (Pareto efficiency) and losers don't pay. Hence RE applies.

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- Another important practical consideration for reserve prices: preventing collusion.

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- Myerson (1981) (Nobel Prize 2007) solves the problem!

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- In case the highest virtual valuation is less than zero, good stays with the seller.
- If bidders are symmetric, we have a second-price auction with a reserve price.

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- One can therefore think of an English auction as a way to “implement” a SP or Myerson-type auction. Start at the reserve price and raise prices until all bidders except one drop out.
- Dynamic auctions are popular in practice - transparent, practical difficulties in collusion.

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- Not known (even for two objects)!
- Selling each good independently may not serve the interests of either efficiency or revenue. Will typically induce complicated strategic behaviour.

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- Yes - by a sealed-bid auction which is a suitable generalization of the SP auction. The VCG auction. Not revenue optimal.
- What is the English auction (dynamic auction) counterpart of the single-good case? Not obvious - active area of research.